PHYSICS/ VECTOR FIELDS : WHAT YOU NEED TO REMEMBER / Dr. Lankar

In this paper, F is a vector, ∇ is a vector, any quantity included in a dot product or in a cross product is a vector.

A vector field on R^3 is a function F(x,y,z) that assigns a vector to every point in space.
F(x,y,z) = <F1,F2,F3> F1 is the x-component of F, F2 is the y-component and F3 the z-component

A scalar field f(x,y,z) is a function that associates a scalar (number) with each point.

dot product = A**.**B = A1B1+A2B2 + A3B3 = |A| |B| cos(angle between A and B) . A and B are vectors.

cross product = A x B = i (Ay – Bz) – j (Ax – Bz) + k (Ax – Bz) = |A| |B| sin (angle between A and B) = surface of the parallelogram A,B. AxB is a vector perpendicular to both A and B. A and B are vectors.

A gradient field of a scalar field (maps scalar to vector)

grad f(x,y,x) = $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ *with* the nabla operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

If z =f(x,y) is the equation of a surface (imagine a mountain) then $\boldsymbol{\nabla f . u}$ measures the rate of change of the function f in the direction u at a given point (xo,yo,zo) . u is a vector in the plan x,y. (see picture website). If all the level curves of f are dropped on the plan x,y , then the vector **∇f at given point is perpendicular to the level curves of f** (f=constant). ∇f at (xo,yo,zo) gives the direction of the largest rate of change. $-\nabla f$ gives the smallest rate of change. Its magnitude is the rate of change.$|\nabla f|$ = largest increase.
$\nabla f . u$ is the component of ∇f along the direction u.

property: $\nabla(u.v) = u.\nabla(v) + v.\nabla(u)$ u and v are vectors.

If z =f(x,y) is the equation of a surface in 3D
then z- zo = f_x (x-xo) + f_y (y-yo) is the equation of the tangent plane to the surface at (xo,yo,zo).
f_x is $\frac{\partial f}{\partial x}$ at (xo,yo) and f_y is $\frac{\partial f}{\partial y}$ at (xo,yo).
The equation of the line tangent to the surface at (xo,yo) is (for z=zo) f_x (x-xo) + f_y (y-yo) =0 .

The components of a normal vector to the surface z= f(x,y) at (xo,yo) is (f_x, f_y, -1) or (-f_x, -f_y, 1) .

If the surface has an equation w=w(x,y,z) then the equation of the plane tangent to the surface at (xo,yo,zo) is (x-xo)f_x + (y-yo)f_y + (z-zo) f_z = 0. the gradient is normal to the level surface.
∇w = normal to surface w(x,y,z)=k

The vector field **F is a gradient field** if it derives from a potential (scalar field). **F = grad(f). F is said to be conservative** . f is called the potential. F is defined over a region R without hole. (In Physics F = - grad(f)). In 2D a gradient field is conservative if $F1_y = F2_x$. In 3D F is conservative if $F1_y=F2_x$, $F1z=F3_x$ and $F2_z=F3_y$ (partial derivatives). Or if **F is conservative then curl F = 0** (curl (grad f) = 0)

If F is conservative then The line integral depends only on the end points P1 and P2 of the curve C. This is true in 2D or 3D : $\int_C F.dr = \int_{P1}^{P2} \nabla f.dr = f(P2) - f(P1)$ according to the fundamental theorem of Calculus.
dr is a vector = dxi+dyj+dz , it belongs to C. (F.dr = F.T dl where T is the unit vector tangent to C and dl a line element of C = magnitude of dr). Also $\oint_C \mathbf{F.dr = 0}$ **work along a close curve = 0**

Note: If the curve is a segment $[AB]$ (in 1D) then the calculus theorem becomes: $\int_A^B f'dx = f(B) - f(A)$
In both case, the theorem connects 1 dimension (line integral) to 0 dimension (points).

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divergence of a vector field in 2D or 3D: (maps vector to scalar) in 2D $\frac{\partial F3}{\partial z} = 0$

$$\text{div(F)} = \nabla . F = \frac{\partial F1}{\partial x} + \frac{\partial F2}{\partial y} + \frac{\partial F3}{\partial z}$$

If F is the velocity of a fluid, in 2D, the divergence measures the amount of fluid crossing a closed curve. In 3D the divergence measures the amount of fluid going leaving a closed surface. For the divergence not to be zero, fluid has be to created (source) or removed (sink) or the density of the fluid has to change. (see below)
in 3D : div(F) = net mass of fluid being created per unit time per unit volume in a volume in 3D.= source rate=amount of flux generated per unit volume. measures the " stretching" component of a field.

$\nabla.(fA) = A.\nabla f$ A is a vector. f is a scalar

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curl of a vector (maps vector to vector)

$$\text{curl(F)} = \nabla \text{xF} = \text{i}\left(\frac{\partial \text{F3}}{\partial \text{y}} - \frac{\partial \text{F2}}{\partial \text{z}}\right) - \text{j}\left(\frac{\partial \text{F3}}{\partial \text{x}} - \frac{\partial \text{F1}}{\partial \text{z}}\right) + \text{k}\left(\frac{\partial \text{F2}}{\partial \text{x}} - \frac{\partial \text{F1}}{\partial \text{y}}\right)$$

If F is a velocity field, the curl measures the angular velocity ($2w$) and its direction. If F is a force field, curl measures the torque exerted by the force per unit moment of inertia . It is a measure of twist in a vector field. curl measures the rotation component in a vector field. In a complicated flow, curl measured how much rotation is taking place in the flow. (if the fluid is rotating around the z-axis V = (-w y, w x,0) , curl V = 2w k, w is the angular velocity, V is the velocity)

The curl measures how mucha vector field " fails to be conservative " curl (grad f))=0.

Laplacian of a scalar filed : $\Delta f = \nabla.\nabla f = \nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ Laplacian is a scalar (maps scalar to scalar)

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properties of vector operators:

$\nabla.\nabla \text{x} F = 0$ or $\text{div(curl F)} = 0$

∇x∇f=0 (curl (grad(f)) =0, grad(f) is a conservative field

∇.(f F) = $\text{f}\,\nabla.\text{F} + \text{F}.\nabla\text{f}$

∇x(fF) = f ∇xF - Fx∇f

$\nabla.(\text{FxG}) = \text{G}.\nabla\text{xF} - \text{F}.\nabla\text{xG}$

∇. (∇f x ∇g) = 0

line integrals: work and flux in 2D

To find a line integral, you need to parameterize C with 1 single variable. You can use t. Example: If C is a circle, then x=cos(t) and y=sin(t) . if C is a parabola $y = x^2$ then x=t and $y=t^2$.

work along a curve C = $\int_C F.dr = \int_C F.T\, dl = \int_C F1\, dx + F2dy$ F.T is the component of F parallel to the curve. T is a unit vector tangent to dl, line element, that belongs to C. The curve C lives in 2D. dr is a vector = dxi + dyj. dl is a line element = magnitude of dr. F is a vector = (F1, F2). If F is gradient field, then the line integral = f(P2) – f(P1) .

Flux of F across C $= \int \mathrm{F.N\, dl} = \int -\mathrm{F2dx + F1dy}$ N is a unit vector normal to the line element dl (dl belongs to C). F.N is the component of F normal (perpendicular) to the curve (at a given point).

To find **double integrals**, you may have to change variable. You need to compute the **Jacobian** :

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} Ux & Uy \\ Vx & Vy \end{vmatrix} \qquad du\, dv = \det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) dx\, dy$$

To compute the surface integral $\iint_S F$.N ds where N is the unit vector normal to the surface S and ds is a area element of S, you need to find Nds and make sure you have only 2 variables in the integral. (parametrize Z with 2 variables). *If S is a plane parallel to x,y (N = +/- k and ds =dxdy) or y,z (N== +/-i ds=dydz) or x,z (N=+/-j ds= dxdz) then it is easy.

* If S is a surface z=f(x,y) then Nds = +/- ($-f_x$, $-f_y$, 1) dxdy

*If you don't know the equation of the surface but you are given N1 a normal vector then N ds = +/- $\frac{N1}{N1.k}$ dx dy

*If the surface is a cylinder of radius a N=+/- $\frac{(x,y,0)}{a}$ and ds = a dz d

*If the surface is a sphere of radius R N=+/- $\frac{(x,y,z)}{R}$ and ds = R^2 sin(Φ) dΦ d

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flux of a vector field in 3D.

The flux is computed now across a membrane (surface) in 3D. If the vector field is a fluid, the flux measures the amount of fluid that crosses the membrane per unit time. Suppose the surface S is oriented by a unit normal vector N. (see picture website). Then :

Flux of F = $\iint_{S=M} F.N\, ds$ S is the membrane through which the flux go through and N is a unit vector normal to the surface element ds (at a given point). F is a vector field.

If z=f(x,y) is the equation of the surface S, then N ds = (f_x,f_y,-1) dx dy N ds is a vector whose magnitude is the surface element ds (in space)/ N is the unit normal vector. dx dy is the shadow of ds on the plane (x,y). ds and dx dy are the areas of a parallelogram.

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work of a vector field in 3D

work = $\int_C F.T\, dl$

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in 2D: **Green Theorem for work : (from line integral to surface integral and vice versa)**

$\iint_{R=A}\left(\frac{\partial F2}{\partial x}-\frac{\partial F1}{\partial y}\right)dA = \oint_{\partial R=\partial A=C} F.dr$ A is the Area surrounded by the oriented curve C (counterclockwise)

$note$: $\left(\frac{\partial F2}{\partial x}-\frac{\partial F1}{\partial y}\right)k \;=\; curl(F)$ with F = F(x,y), no z variable. $F.dr = F.T\ dl$ dr is a vector,. (the magnitude of dr is dl a line element of C at a given point). F.T is the component of F along C at a given point. dr = dx i + dy j

in2D: **Green Theorem for flux: (from line integral to surface integral and vice versa)**

$\iint_{R=A}\left(\frac{\partial F1}{\partial x}+\frac{\partial F2}{\partial y}\right)dA = \iint_A div(F)dA = \oint_{\partial R=\partial A=C} F.N\ dl$

F.N is the component of F perpendicular to the curve C (at a given x,y point). N is a unit vector normal to the line element dl that belongs to the curve C at a given point. dl is a scalar (magnitude of dr = dx i + dy j)

NOTE: The green theorem is a generalization of the theorem of calculus. It connects double integral to a line integral. $\int_{\boldsymbol{R}} \boldsymbol{dw} = \int_{\boldsymbol{\partial R}} \boldsymbol{w}$ R is a region and ∂R is the boundary of R. Here R is a surface in 2D and ∂R the closed curve around it.

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in 3D, the **Green theorem** for work becomes the **stoke's theorem :**

$\iint_{R=M=S} \nabla \text{x} F\ .Nds = \iint_{R=M} curl(F)\ .dS = \oint_{\partial R=\partial M=C} F.dr$

M is now a membrane in 3D (or surface S) , C is the boundary of M = curve in 3D. The integral of the curl of a vector field over a surface in 3D equals the line integral of the vector field over the closed curve bounding the region. N ds is a vector. N is the unit normal vector perpendicular to the surface element ds at(x,y,z).

Triple integrals

$\iiint_V f\ dv$ You integrate a scalar quantity. dV = dxdydz or in cylindrical coordinates dV=dz r dr d□ or in spherical coordinates dV = R^2 dR sin(Φ) dΦ d□

Applications = finding the mass, average value of a function, center of mass, moment of inertia…..

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in3D, the Green theorem for flux becomes **the divergence theorem:**

$\iiint_{R=V} div(F) = \iiint_{R=V} \nabla.F\ dV = \oiint_{\partial R=\partial V=M} F.NdS$ The region is a volume V (scalar) , the boundary is the membrane M around the volume. The integral of the divergence of a vector field over some solid equals the integral of the flux through the closed surface bounding the surface. N dS is a vector.

NOTE: This theorem connects double to triple integral. The divergence theorem is a generalization of the theorem of calculus. It connects double integral to a line integral. $\int_R \boldsymbol{dw} = \int_{\partial R} \boldsymbol{w}$ R is a volume V and ∂R is the boundary of $V =$ closed surface.

Using the divergence theorem and some Physics we get the **diffusion equation** that governs how the concentration u (density) of a fluid (smoke in air, dye in water, temperature) changes with time and position. u is the function concentration. $\boxed{\frac{\partial u}{\partial t} = k\, \nabla^2 u = k\, div(\nabla u)}$

proof: with F is a vector field and if F the velocity of the flux then F = -k (∇u) and the rate of change of u = amount of fluid leaving = flux of F = div(F) so div(F) = - $\frac{\partial u}{\partial t}$. (we suppose no source or sink)

This can also be written as $\boxed{\frac{\partial u}{\partial t} = \text{div (F)} + \Psi}$ F is the velocity of the fluid times the density u (F=uv, v is the velocity of the fluid and u is the density) , Ψ is the source density minus the sink density. This is also called the **equation of continuity.**
