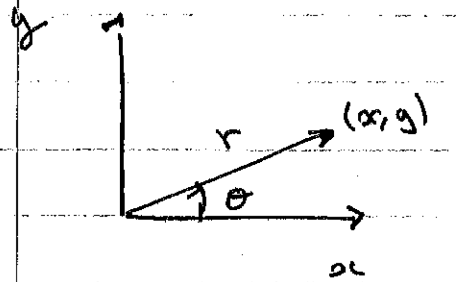
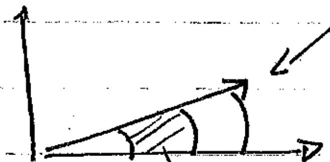
double integrals - polar coordinates
Change of variables - Jacobians.



$x = r\cos\theta$
$y = r\sin\theta$.

| Slice = fix θ and run r.

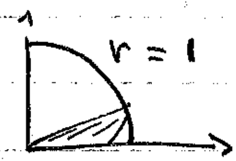


$dr\,(r\,d\theta) = r\,dr\,d\theta$

$dA = r\,dr\,d\theta$ in polar coordinate.

Example:

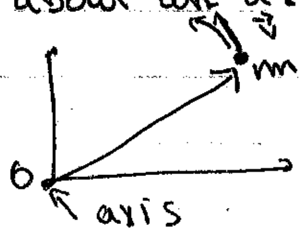
$$I = \iint (1-x^2-y^2)\,dA$$



$$I = \int_0^{\pi/2} d\theta \int_0^1 (1-r^2)\,r\,dr \quad \text{with } dA = r\,d\theta\,dr$$

$$I = \int_0^{\pi/2} d\theta \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{8}$$ easier this way.

Example 2: moment of inertia = resistance to rotation motion $\Rightarrow$ how hard it is to rotate about an axis.

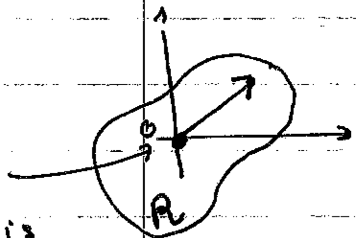


point mass $\quad V = r\omega$

Kinetic energy $= \frac{1}{2} m r^2 \omega^2$

so moment of inertia for point mass $= m r^2$

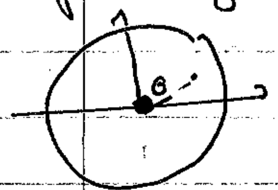
consider a solid with density δ (per surface)



$$\underbrace{I_0}_{\substack{\text{moment}\\ \text{of inertia}\\ \text{about } O}} = \iint_R r^2\,\delta\,dA.$$

axis
$\perp$ to paper

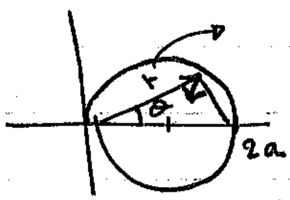
Example: Find the moment of inertia of a disk, spinning about O, radius a. $\delta = 1$



$$\int_0^{2\pi}\int_0^{a} r^2\, r\,d\theta\,dr = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^{a} d\theta$$

$$= \frac{2\pi a^4}{4} = \frac{a^4\pi}{2}$$

Example: Now the axis is placed at the edge of the disk.



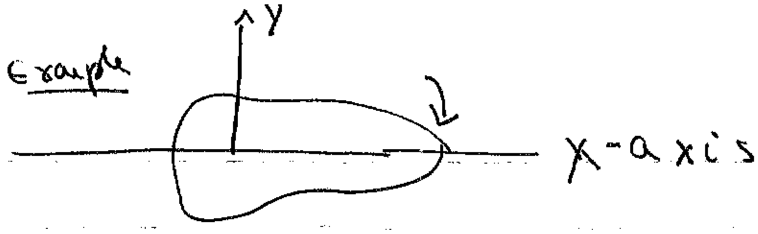
~~[illegible]~~

if θ is fixed,
r runs from 0 to $2a\cos\theta$

$$\int_{-\pi/2}^{\pi/2}\cancel{\pi}\,d\theta\int_0^{2a\cos\theta}\cancel{d\theta}\; r^3\,dr = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2a\cos\theta} = \int_{-\pi/2}^{\pi/2}\frac{16a^4\cos^4\theta}{4}\,d\theta$$

$$= 4a^4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = \boxed{\frac{3}{2}\pi a^4}$$

Example



$$I = \iint y^2 \delta \, dA.$$

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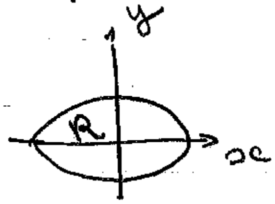
change of variables – Jacobians.

Example 1 find area of an ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

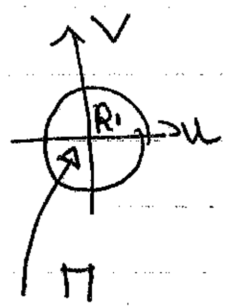


$$A = \iint_R dx\,dy$$

$u = \frac{x}{a} \Rightarrow dx = du\,a$

$v = \frac{y}{b} \Rightarrow dy = dv\,b$

$$A = ab \iint_{u^2+v^2 \leq 1} du\,dv$$ the region becomes a circle of radius 1



so $A = ab\pi$.

the scaling factor between R and R1 is ab.

Example 2 $\begin{cases} u = 3x - 2y \\ v = x + y \end{cases}$ we need linear approximation to find $\Delta u \, \Delta v$

$$\begin{cases} \Delta u = u_x \Delta x + u_y \Delta y \\ \Delta v = v_x \Delta x + v_y \Delta y \end{cases}$$

so $\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$

linear approximation:

$$\Delta u \Delta v \simeq (u_x v_y - u_y v_x)\Delta x \Delta y$$

$$\Delta u \Delta v = \det\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \Delta x \Delta y$$

or $\Delta u \Delta v = \frac{\partial(u,v)}{\partial(x,y)} \Delta x \Delta y$ ← Jacobian

and $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

Example1: polar coordinates.

$x = r\cos\theta$ $\quad$ $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$y = r\sin\theta$

$J = r\cos^2\theta + r\sin^2\theta = r$

$$\boxed{\Delta x \Delta y = r \Delta\theta \Delta r}$$

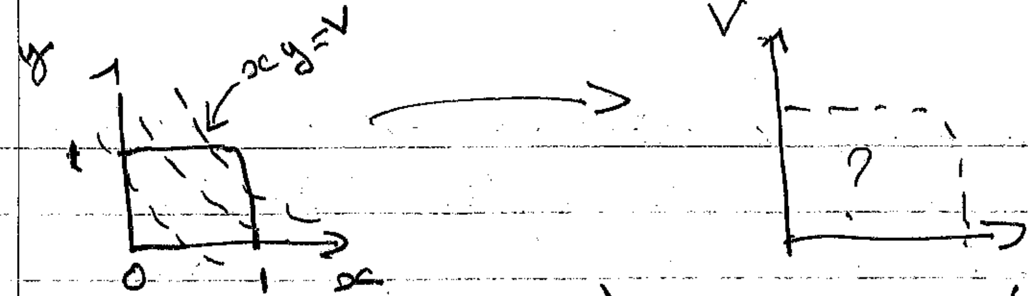
Example2 compute $\int_0^1\int_0^1 x^2 y \, dx\, dy$ by changing

variable $\begin{cases} u = x \\ v = xy \end{cases}$ $\quad$ $\left\{ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x \right.$

$\Delta u \Delta v = x \Delta x \Delta y \Rightarrow \Delta x \Delta y = \frac{1}{x}$

$I = \iint_{?} uv \, du\, dv$ $\quad$ we need to find the bounds.

1st way



if V = constant xy = constant = hyperbola

$0 < V < 1$, along the hyperbola u varies.

if $\underbrace{y = 1, \ u = V}_{\text{hyperbola enters the region}}$ if ~~given~~ $\underbrace{x = 1, \ u = 1}_{\text{hyperbola leaves the region}}$

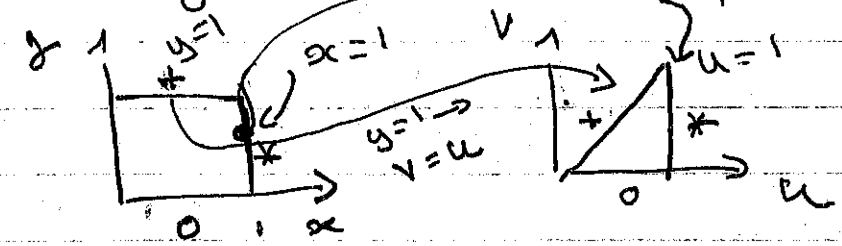
so $V < u < 1$
$0 < V < 1$

$$I = \int_0^1 V \int_V^1 du \, dV$$

$$I = \int_0^1 V \Big[u\Big]_V^1 dV = \int_0^1 V(1-V)\,dV$$

$$I = \left[\frac{V^2}{2} - \frac{V^3}{3}\right]_0^1 = \frac{1}{6}$$

2nd way to find bounds. the line $x=1$ becomes $u=1$



same answer.