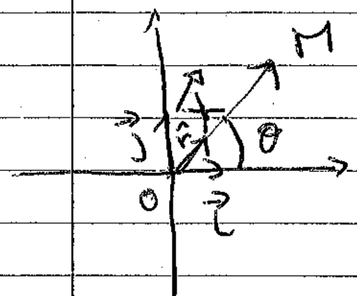
Kepler laws

I * 1st law = orbits are in a plane.

Notation:



$$\vec{OM} = \vec{X} = r\hat{r}$$

$$\vec{V} = \frac{d\vec{X}}{dt} = \frac{d\vec{r}}{dt} \qquad \vec{a} = \frac{d^2\vec{X}}{dt^2} \text{ acceleration}$$

$$\vec{L} = m\vec{X} \times \vec{V} \quad \text{: angular momentum}$$

$$\vec{J} = \vec{X} \times \vec{V} \quad \text{: angular momentum per unit mass}$$

In coordinnate polar:

$$\vec{X} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j}$$

$$\vec{V} = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\vec{i} + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})\vec{j}$$

$$\boxed{\vec{J} = \vec{X} \times \vec{V}} = r^2\dot{\theta}\,\vec{k} \Rightarrow \text{motion in } (x,y) \text{ plane.}$$

II - orbit = ellipse

* equations we will use.

$$(1) \quad \boxed{\vec{J} = \vec{X} \times \vec{V}} \qquad (1) \text{ into } 2$$

$$\|\vec{X}\cdot\vec{V}\|^2 + \|\vec{X} \times \vec{V}\|^2 = \|X V\|^2$$

$$\boxed{\|\vec{X}\cdot\vec{V}\|^2 + J^2 = r^2 V^2} \quad (2)$$

* $\vec{X}\cdot\vec{X} = r^2$

$$(\vec{X}\cdot\vec{X})' = (r^2)' \Rightarrow \vec{X}'\cdot\vec{X} = rr'$$

$$\Rightarrow \boxed{\vec{V}\cdot\vec{X} = rr'} \quad (3)$$

(3) into 2 $\quad (rr')^2 + J^2 = r^2V^2$

$$(4) \quad \boxed{(r')^2 + \frac{J^2}{r^2} = V^2}$$

* derive (3)

$$\vec{V}'\cdot\vec{X} + \vec{V}\cdot\vec{X}' = rr'' + (r')^2$$

$$\cancel{a\cdot\vec{X} + V\cdot V = (rr'') + (r')^2}$$

$$(5) \quad \boxed{\vec{a}\cdot\vec{X} + V^2 = (rr'') + (r')^2}$$

(4) into (5) $\quad \vec{a}\cdot\vec{X} + \cancel{(r')^2} + \dfrac{J^2}{r^2} = (rr'') + \cancel{(r')^2}$

$$(6) \quad \boxed{\vec{a}\cdot\vec{X} + \frac{J^2}{r^2} = (rr'')}$$

* Newton 2nd Law + gravitationnal law

$$\vec{F} = -\frac{GmM}{r^2}\hat{r} \qquad \vec{F} = m\vec{a}$$

$$\boxed{\vec{a} = -\frac{GM}{r^2}\hat{r}} \quad (7)$$

(7) into (6)

$$\boxed{-\frac{GM}{r^2}r + \frac{J^2}{r^2} = (rr'')} \quad (8)$$

change of variable $u(\theta) = \frac{1}{r}$ $\quad u' = \frac{du}{d\theta}$

$\cancel{\frac{du}{d\theta}\frac{d\theta}{dt}}$ $\left\{ r = \frac{1}{u} \quad r' = -\frac{u'}{u^2}\frac{d\theta}{dt} = -r^2 u'\dot{\theta} \right.$

$$= -Ju'$$

(9) $\quad r'' = -J\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{J^2}{r^2}\frac{d^2u}{d\theta^2} = -J^2\frac{d^2u}{d\theta^2}u^2$

(9) into (8) $\quad -\frac{GM}{r} + \frac{J^2}{r^2} = r\left(-J^2\frac{d^2u}{d\theta^2}\cdot u^2\right)$

$$-GMu + J^2u^2 = -uJ^2\frac{d^2u}{d\theta^2}$$

(10) $\quad +GM = J^2u + J^2\frac{d^2u}{d\theta^2}$

or (10) $$\frac{d^2u}{d\theta^2} + u = \frac{GM}{J^2}$$

2nd degree linear equation

if $\frac{d^2u}{d\theta^2} + u = 0$ then $u = A\cos\theta$

So $$\boxed{u(\theta) = A\cos\theta + \frac{GM}{J^2}} \quad (11)$$

$$u(\theta) = \frac{GM}{J^2}\left(\frac{AJ^2}{GM}\cos\theta + 1\right)$$

$$\boxed{r = \frac{J^2/GM}{\frac{AJ^2}{GM}\cos\theta + 1}}$$

ellipse equation in polar coordinnates with $e = \frac{AJ^2}{GM}$ and $p = 1/A$

III The angular momentum is constant

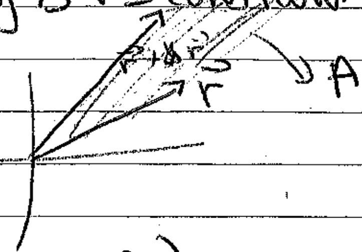
$$\vec{L} = m\vec{X} \times \vec{V}$$

$$\frac{d\vec{L}}{dt} = \underbrace{m\vec{V} \times \vec{V}}_{0} + \underbrace{m\vec{X} \times \frac{d\vec{V}}{dt}}_{0}$$

because $\frac{d\vec{V}}{dt} = -\frac{GM}{r^2}\hat{r}$

~~d~~ So $\boxed{\vec{L} = \text{constant}}$

III The area swept during Δt = constant.



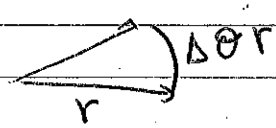
$$\Delta \text{Area} = \frac{1}{2}\left(\vec{r} \times \Delta\vec{r}\right)$$

$$\frac{\Delta \text{Area}}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}\right) \xrightarrow[\Delta r \to 0]{} \frac{1}{2}\left(\vec{r} \times \vec{V}\right)$$

$$\boxed{\frac{d\,\text{Area}}{dt} = \frac{1}{2M}\vec{L}} = \text{constant}$$

The change in area is constant with time.

or:



$$\Delta \text{Area} = \frac{r^2 \Delta\theta}{2}$$

$$\frac{\Delta \text{Area}}{\Delta t} = \frac{r^2}{2}\frac{\Delta\theta}{\Delta t} = \frac{L}{2M}$$

V total energy determines the conic section.

$E_K = \frac{1}{2} M V^2$ (Kinetic energy)

From previous: (4)

$$(r')^2 + \frac{J^2}{r^2} = V^2$$

let's find r'

$$r' = \frac{J^2}{GM_0}\left\{\left(\frac{1}{(1+e\cos\theta)}\right)'\right\} = \frac{J^2}{GM_0}\left(\frac{+e\sin\theta}{(1+e\cos\theta)^2}\dot{\theta}\right)$$

$$= \frac{J^2}{GM_0}\left(\frac{e\sin\theta}{(1+e\cos\theta)^2}\frac{J}{r^2}\right) = \frac{J^2}{GM_0}\frac{\sin\theta J}{(1+\cos\theta e)^2} \times \frac{(1+e\cos\theta)^2}{J^4/(GM_0)^2}$$

(11) $r' = \frac{e\sin\theta \cancel{J}}{J^2/(GM_0)} \Rightarrow (r')^2 = \frac{e^2\sin^2\theta (GM_0)^2}{J^2}$

(11) into (4)

(12) $$V^2 = \frac{e^2\sin^2\theta (GM_0)^2}{J^2} + \frac{\cancel{J^2} \times (1+e\cos\theta)^2}{J^{\cancel{4}2}/(GM_0)^2}$$

$$V^2 = \frac{e^2\sin^2\theta (GM_0)^2}{J^2} + \frac{(1+e\cos\theta)^2 GM_0}{J^2}$$

$$\text{total energy} = \frac{1}{2} M V^2 - \frac{G M M_0}{r}$$

$$= \frac{1}{2} M \frac{(G M_0)^2}{J^2} \left(e^2 \sin^2\theta + 1 + 2e\cos\theta + e^2 \sin^2\theta \right)$$

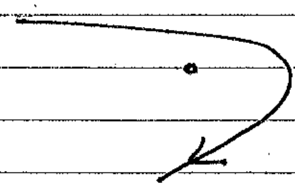
$$- G M M_0 \left(\frac{G M_0}{J^2} \right) (1 + e\cos\theta)$$

$$= \frac{M (G M_0)^2}{J^2} \left(\frac{e^2 \sin^2\theta}{2} + \frac{1}{2} + \frac{2e\cancel{\cos\theta}}{\cancel{2}} + \frac{e^2 \sin^2\theta}{2} - 1 - \cancel{e\cos\theta} \right)$$

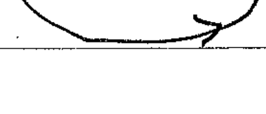
$$= \frac{M (G M_0)^2}{J^2} \left(\frac{e^2}{2} - \frac{1}{2} \right) = \boxed{\frac{M (G M_0)^2 (e^2 - 1)}{2 J^2}}$$

Total energy of object in orbit: $\frac{1}{2} \frac{M (G M_0)^2}{J^2} (e^2 - 1)$

So if $e = 1$ total energy $= 0$ the kinetic energy balances the potential energy, the conic is a parabola. The object can escape the gravitational attraction.



* if $0 \leq e < 1$ total energy < 0 the object describes an ellipse. the object does not enough energy to escape.



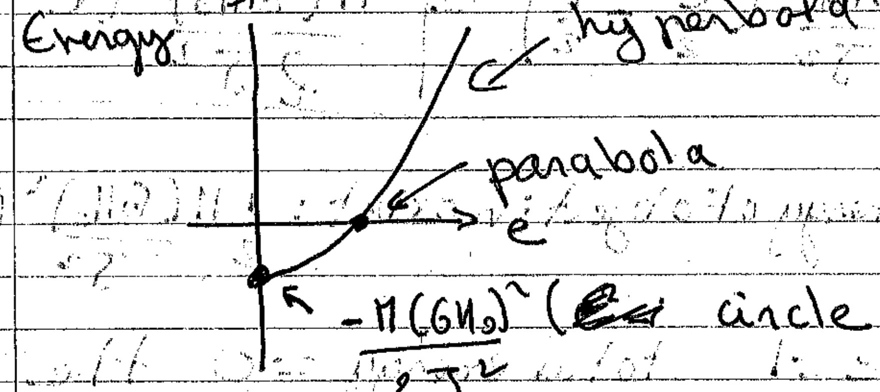
$e = 0$ it's a circle $\Rightarrow$
lowest energy state $= -\frac{M (G M_0)^2}{2 J^2}$

$e > 1$ it's a hyperbola

total energy > 0: Kinetic energy > potential energy.

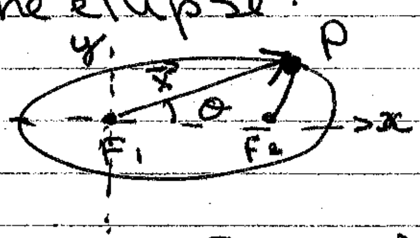
total energy $\propto (e^2 - 1)$

interplay between K and U



VI equation of a conic is $\begin{cases} r = \dfrac{ep}{1 \pm e\cos\theta} \\ r = \dfrac{ep}{1 \pm \sin\theta} \end{cases}$

let's use the ellipse.



$r = |PF_1|$

$d = |PF_2|$

$\|F_1F_2\| = 2c \qquad F_2(2c,0) \qquad F_1(0,0)$

ellipse $\Rightarrow \|F_1P\| + \|F_2P\| = 2a$

$$r + \|r - 2c\vec{i}\| = 2a$$

$\|r - 2c\vec{i}\| = (2a - r)$

$\cancel{r^2} - \cancel{4} c\cos\theta r + \cancel{4} c^2 = \cancel{4} a^2 - \cancel{4} ar + \cancel{r^2}$

$$c^2 - a^2 = r(c\cos\theta - a)$$

$$r = \frac{c^2 - a^2}{c\cos\theta - a} = \frac{+b^2}{c\cos\theta - a} \quad \text{with } c^2 - a^2 = b^2$$

$\boxed{r = \dfrac{b^2/a}{\frac{c}{a}\cos\theta - 1}}$ with $e = \dfrac{c}{a}$ eccentricity