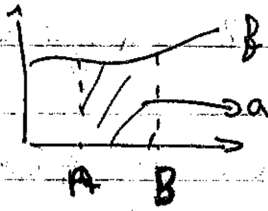
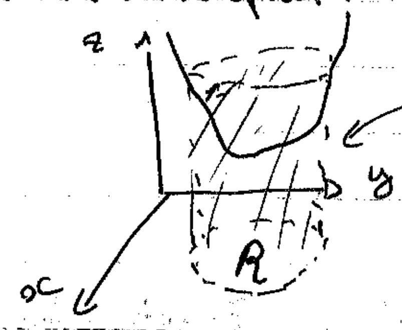
double integrals

- single integral $\int_A^B f(x)\,dx$ = area below the graph

integral over $[A,B]$



- double integral = volume below the graph $z = f(x,y)$



volume below graph
integration over a region R
in the X – Y plane.

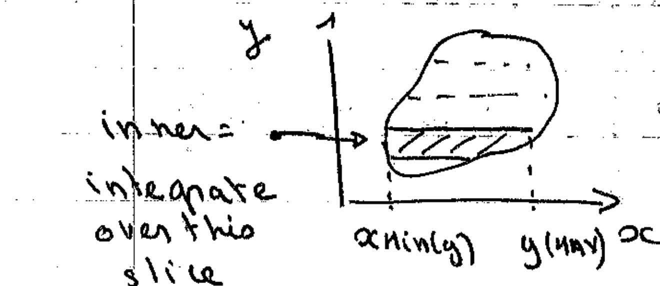
$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

Big Idea:

called interation.

inner integral $\displaystyle\int_{x_{min}(y)}^{x_{max}(y)} f(x,y)\,dx$ } Volume of 1 slice for y constant $V_{slice}(y)$

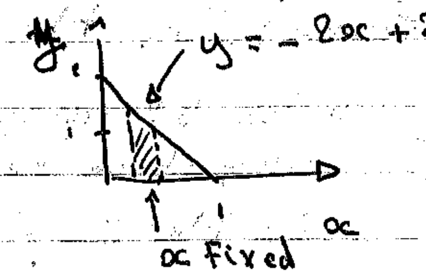
outer integral : $\displaystyle\int_{y_{min}}^{y_{max}} V_{sice}(y)\,dy$ = total volume.



you can also keep x constant first and get vertical slice.

Example 1: Find the volume of the solid below the plane $z = 1 + y$, bounded by the coordinates planes and the vertical plane $2x + y = 2$

Solution: $z = f(x,y)$. 1st step = draw the region, you are integrating on.



then decide if

you slice the volume vertically (x is fixed, and y varies for inner integral) or if you slice horizontally (y is fixed and x runs for inner integral.

let's keep x fixed first

$$\text{Volume} = \iint_R f(x,y)\,dy\,dx = \int_0^1 dx \int_{y_{min}=0}^{y_{max}=-2x+2} (1+y)\,dy$$

Inner integral = volume of slice $S(x)$

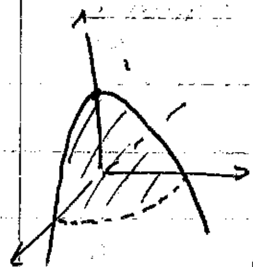
$$= \int_0^{-2x+2} (1+y)\,dy = \left[y + \frac{y^2}{2}\right]_0^{-2x+2} = 4x - 3x^2 + \frac{2}{3}x^3$$

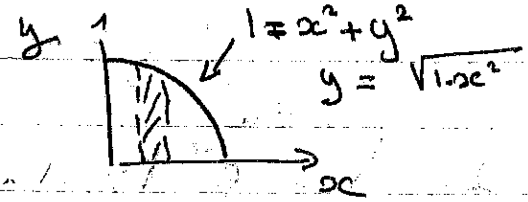
$$\text{Volume} = \int_0^1 4x - 3x^2 + \frac{2}{3}x^3\,dx = \frac{5}{3}$$

Example 2: Find the volume below $z = 1 - x^2 - y^2$ (paraboloid open down and shifted up of 1).

above the plane (x, y)

solution: Region $\mathbb{R}$:





Inner: volume of $S(x) = \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy = \frac{2}{3}(1 - x^2)^{3/2}$

$$\text{Volume} = \int_0^1 \frac{2}{3}(1 - x^2)^{3/2}\, dx$$

change of variable $\quad x = \cos\theta \quad dx = -\sin\theta\, d\theta$

$x = 0,\ \theta = \pi/2$ and $x = 1,\ \theta = 0$

$$\text{volume} = -\frac{2}{3}\int_{\pi/2}^{0} \sin^3\theta \sin\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \Rightarrow \sin^4\theta = \frac{(1 - \cos 2\theta)^2}{4}$$

$$\text{Volume} = \frac{1}{6}\int_0^{\pi/2} (1 - \cos 2\theta)^2\, d\theta = \frac{1}{6}\int_0^{\pi/2} 1 - 2\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$\cos^2 2\theta = \frac{1}{2}(1 + \cos 4\theta)$$

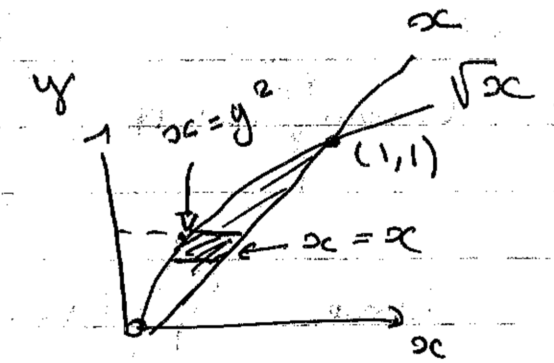
$$\text{volume} = \frac{1}{6}\int_0^{\pi/2} 1 - 2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\, d\theta = \frac{1}{6}\left[\theta + \frac{2\sin 2\theta}{2} + \frac{\theta}{2} - \frac{1}{8}\sin 4\theta\right]_0^{\pi/2}$$

$\text{volume} = \frac{\pi}{8}$

Example 3 : sometimes, you should change the order of integration to get easier integrals.

$$\to I = \int_0^1 \underbrace{\int_x^{\sqrt{x}} \frac{e^y}{y}\, dy}_{\text{inner}}\, dx$$

inner integral hard because of $\frac{e^y}{y}$



let's change the order of integration

$$I = \int_0^1 \frac{e^y}{y}\, dy \int_{y^2}^{y} dx = \int_0^1 \frac{e^y}{y}(y - y^2)\, dy$$

$$= \Big[-ye^y + 2e^y \Big]_0^1 = \boxed{e - 2}$$

Application ① Mass $\Delta m = \delta \Delta A$ } δ is density = mass per unit area.

$\text{Mass} = \iint_R \delta\, \Delta A$

② Average value of f in a Region R $\bar{f} = \frac{1}{\text{Area}} \iint f\, dA$ equally likely

③ weighted Average $= \frac{1}{\text{Mass}(R)} \iint_R f \delta\, dA$

④ Center of mass $\bar{x} = \frac{1}{\text{Mass}} \iint_R x \delta\, dA$ and $\bar{y} = \frac{1}{\text{Mass}} \iint_R y \delta\, dA$.