<u>linear 2nd order diff equ</u>
<u>homogeneous</u>

Standard Form:

$$y'' + Ay' + By = 0 \quad \text{(homogeneous form)}$$

A and B are real and constant.

Let's assume the general solution is

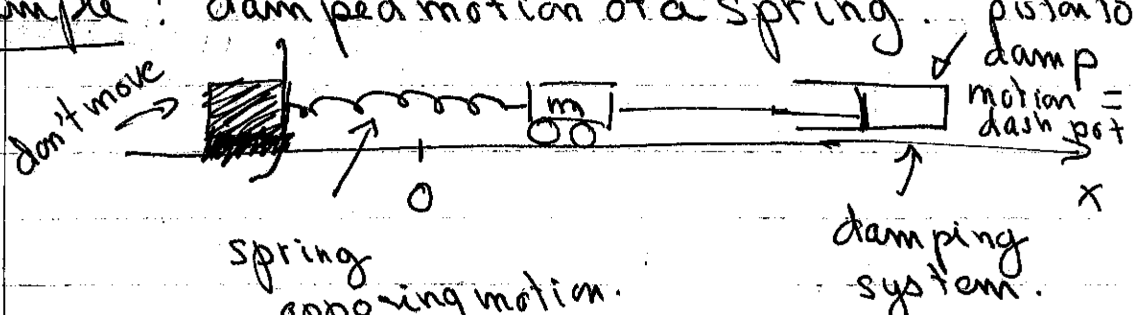
$$\boxed{y = C_1 y_1 + C_2 y_2}$$

y_1 and y_2 are independent ($y_1 \neq K y_2$)

C_1 and C_2 are arbitrary constant defined by initial conditions.

<u>Note</u> order of equation = # of constant to find.

<u>Example</u>: damped motion of a spring.



Newton 2nd law:

$$m\ddot{x} = \underbrace{-k x}_{\text{Hookes Law Force applied by spring on mass}} \underbrace{-c\dot{x}}_{\text{Force applied by damping system = depends on the speed}}$$

k is the spring constant
C is the damping constant

the ratio k/c will determines the damping of the motion ⇒ iF ~~the~~ the mass will oscillate or no.

we need to find $y = C_1 y_1 + C_2 y_2$.
suppose $y = e^{rt}$ and let's plug it in equation:

$$x'' + \frac{C}{m} x' + \frac{k}{m} x = 0$$

$$r^2 e^{rt} + \frac{C}{m} r e^{rt} + \frac{k}{m} e^{rt} = 0$$

$$\boxed{r^2 + \frac{c}{m} r + \frac{k}{m} = 0}$$

this is called the characteristic equation.

3 cases
1. roots are real and distinct
2. pairs of conjuguate numbers
3. real and equal. (double solution)

$\boxed{\text{case 1}}$ we have 2 real solutions r_1 and r_2
so we have the 2 solutions y_1 and y_2.

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Example case 1

more damping.

$$r^2 + 4r + 3 = 0$$
$$(r+3)(r+1) = 0$$
$r = -1$ and $r = -3$

{ damping > spri constant

$$y = C_1 e^{-3t} + C_2 e^{-t} \qquad r = -1, \quad r = -3$$

initial conditions.

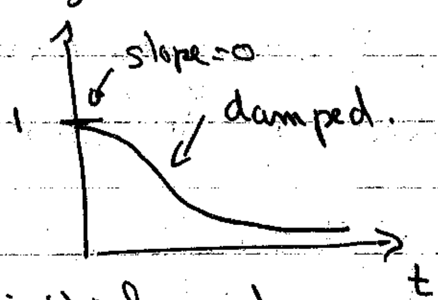
more m by 1 $\to$ $y(0) = 1$ $\to$ $1 = C_1 + C_2$

realeased From rest $\to$ $y'(0) = 0$ $\qquad 0 = -3C_1 - C_2 = 0$

$$1 = -2C_1 \Rightarrow C_1 = -1/2$$
$$C_2 = 1 - C_1 = 3/2$$

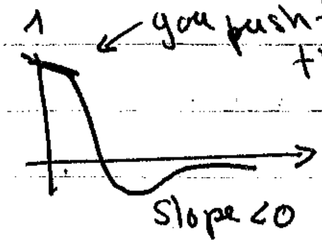
$$y = -\frac{1}{2} e^{-3t} + \frac{3}{2} e^{-t}$$

$y(0) = 1 \quad y'(0) = 0 \Rightarrow$ slope $= 0$ (Flat)

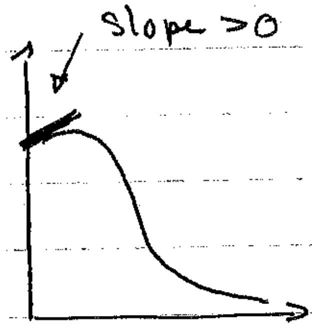


No initial speed.

If $y'(0) \neq 0$, you push the mass (in the negative direction.), it has time to go back and Fath



If initial speed is positive, you push the mass toward the damping part of the system. the mass can't go back toward the spring.



$\boxed{\text{case 2}}$ the roots are complex $r = a \pm bi$

so $\underbrace{\tilde{y}}_{\text{complex}} = e^{(a+bi)t} = e^{at} e^{ibt}$ How to find the real solutions?

$y_1 = \mathbb{R}e(\tilde{y})$ and $y_2 = Im(\tilde{y})$

so $y = C_1 e^{at}(\cos bt) + C_2(\sin bt) e^{at}$

$$\boxed{y = e^{at}(c_1 \cos bt + C_2 \sin bt)}$$

* proof if $u + Vi$ is solution of $y'' + Ay' + By = 0$
then $(u+Vi)'' + A(u+Vi)' + B(u+Vi) = 0$
$\Rightarrow u'' + Au' + Bu + i(V'' + AV' + BV) = 0$
$\Rightarrow u'' + Au' + Bu = 0$ and $V'' + AV'' + BV = 0$
so Real part and Imaginary part are solutions.

Example of case 2

Now damping constant < spring constant.

$y'' + 4y + 5 = 0$

characteristic equation: $r^2 + 4r + 5 = 0$

quadratic equation: $r = \frac{-4 \pm 2i}{2} = -2 \pm i$

let's choose 1 root $r = -2 + i$

(you get the same thing if you take $-2-i$)

$$\tilde{y} = e^{-2t} e^{it} \Rightarrow y_1 = e^{-2t}\cos t \quad y_2 = e^{-2t}\sin t$$

so $y = C_1 e^{-2t}\cos t + C_2 e^{-2t}\sin t$

$y(0) = 1 \Rightarrow C_1 = 1$

$y'(0) = 0 \Rightarrow C_2 = 2C_1 = 2$

so $y = e^{-2t}(\cos t + 2\sin t)$ let's add $\cos t + 2\sin t$.

representation of Fresnel to add $\cos t + 2\sin t$

$$\cos t + 2\sin t = \cos t + 2\cos\left(t - \frac{\pi}{2}\right) =$$

$$= \mathrm{Re}\left(e^{it} + 2e^{i(t-\pi/2)}\right) = \mathrm{Re}\left(e^{it} + 2e^{it}e^{-\pi/2}\right)$$

$$\mathrm{Re}\left(e^{it}\left(1 + 2e^{-\pi/2}\right)\right)$$

$$= \mathrm{Re}\left(e^{it}\sqrt{5}\, e^{-i\phi}\right)$$

$$= \mathrm{Re}\left(\sqrt{5}\, e^{i(t-\phi)}\right) = \sqrt{5}\cos(t-\phi)$$