Solving $y' + P_2 y = P_2 q(t)$ when $q(t) = c\cos\omega t$.

Complexification of a linear differential (1st order)

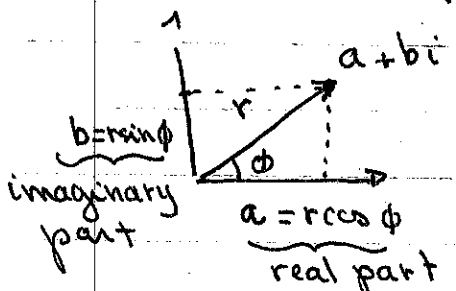
I) Review complex numbers:

- $z = a + bi$ $\quad$ $\bar{z} = a - bi$ $\quad$ $z\bar{z} = a^2 + b^2$

  z is a complex number $\quad$ $\bar{z}$ is the conjugate

- $\dfrac{2+i}{1-3i} = \dfrac{2+i}{1-3i} \cdot \dfrac{1+3i}{1+3i} = \dfrac{-1+7i}{1+9} = -\dfrac{1}{10} + \dfrac{7}{10} i$

- Euler introduced exponential (using polar representation) to write complex number.



$$a + bi = r\cos\phi + i r\sin\phi$$

$$a + bi = r(\cos\phi + i\sin\phi)$$

$$\boxed{a + bi = r e^{i\phi}}$$

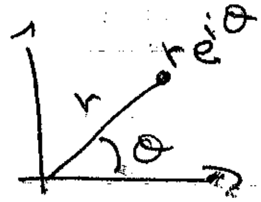
r is the modulus $\quad$ ϕ is the argument

$\tan\phi = \dfrac{b}{a}$, real

we can do that because ① $e^{i\phi_2} e^{i\phi_2} = e^{i(\phi_2 + \phi_2)}$

② $\dfrac{de^{i\phi}}{d\phi} = i\, e^{i\phi}$

the polar form becomes $\longrightarrow$



- $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$

then $z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\underbrace{\underline{\text{modulus}(z_1 z_2)}}_{\underbrace{|z_1 z_2|}_{r_1 r_2}} = \underbrace{|z_1|}_{r_1} + \underbrace{|z_2|}_{r_2} = r_1 + r_2$

- $z = r e^{i\phi} \longrightarrow \frac{1}{z} = \frac{1}{r} e^{-i\phi}$

- Example of application for derivation:

$$\int e^{-x} \cos x \, dx = \mathbb{R}e \int e^{-x} e^{ix} dx$$

$$\int e^{-x} e^{ix} = \int e^{x(i-1)} dx = \frac{1}{i-1} e^{x(i-1)}$$

$$= \frac{i+1}{2} e^{x(i-1)} = \frac{i+1}{2} \{\cos x + i \sin x\} e^{-x}$$

$$\mathbb{R}e = \boxed{\frac{e^{-x}}{2}(\cos x - \sin x)}$$

- $\sqrt[n]{1}$ has n roots in the complex numbers syst.

$$\sqrt[5]{1} = e^{i\frac{2\pi}{5}} = \cos\left(\frac{2\pi}{5}\right) + i\cos\left(\frac{2\pi}{5}\right)$$

$, \frac{2\pi}{5}, \frac{6\pi}{5}, \frac{4\pi}{5}, \frac{8\pi}{5}, 2\pi \Big\}$ are solutions let's solve $y' + ky = kq(t)$ if $\underbrace{q(t) = \cos\omega t}_{\text{input in system.}}$

we complexify the equation:

$$\tilde{y}' + k\tilde{y} = k e^{i\omega t}$$

the solution

y is the real part of $\tilde{y} = y_1 + y_2 i$ ↳ solution.

the integrating factor is $e^{kt} \Rightarrow$

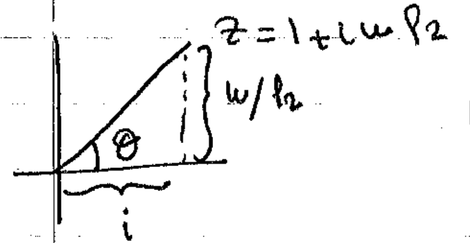
$$(\tilde{y} e^{kt})' = k\, e^{kt} e^{i\omega t}$$

$$(\tilde{y} e^{kt})' = k\, e^{(k+i\omega)t} \underset{\text{integrate}}{\Rightarrow} \tilde{y} e^{kt} = \frac{k}{k+i\omega} e^{(k+i\omega)t}$$

$$\tilde{y} = \frac{k e^{i\omega t}}{k+i\omega} \Rightarrow \tilde{y}(t) = \frac{e^{i\omega t}}{1 + i\omega/k}$$

we need to find the Real part of $\tilde{y}$.

$$\frac{1}{z} = \frac{1}{1 + i\omega/k} \qquad \arg(z) = -\arg\left(\frac{1}{z}\right)$$



$\theta = \arg(z)$

$\arg\left(\frac{1}{z}\right) = -\theta$

$$\left|\frac{1}{z}\right| = \frac{1}{\sqrt{1+\omega^2/k^2}}$$

so

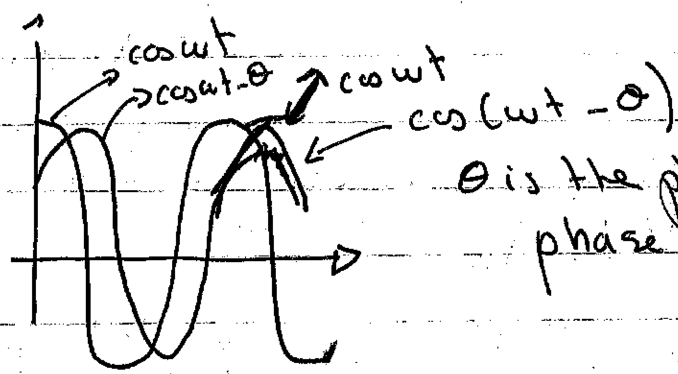
$$\frac{1}{1+i\omega/k} = \frac{1}{\sqrt{1+\frac{\omega^2}{k^2}}}\, e^{-\theta i}$$

with $\tan\theta = \omega/k$

$$\tilde{y}(t) = \frac{1}{\sqrt{1+\omega^2/k^2}}\, e^{i\omega t - i\theta} = \frac{1}{\sqrt{1+\omega^2/k^2}}\, e^{i(\omega t - \theta)}$$

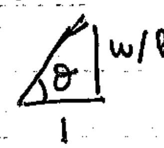
$$\boxed{y = \mathrm{Re}\left(\tilde{y}(t)\right) = \frac{1}{\sqrt{1+\omega^2/k^2}} \cos(\omega t - \theta)}$$ solution = response to $\cos \omega t$

$\cos(\omega t - \theta)$ is $\cos \omega t$ shifted θ units @ right



θ is the phase lag or phase delay.

if $k \uparrow$ $A \to 1$ $\theta \to 0$, the response is close to the input



if $k \searrow$ the solution is damped.