Curl and Green theorem.

Definition: curl $\vec{F} = N_x - M_y$

So if $\vec{F}$ is conservative curl $\vec{F} = 0$

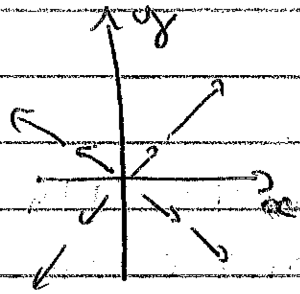
because $N_x = M_y$.

curl measures the angular velocity ω if the field is a vector field (curl $\vec{V} = 2\omega$)

curl measures the torque if the field is a force field.

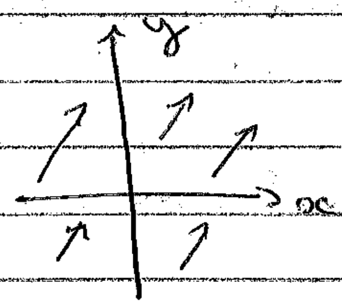
Measures the rotation component of a motion.

Example:



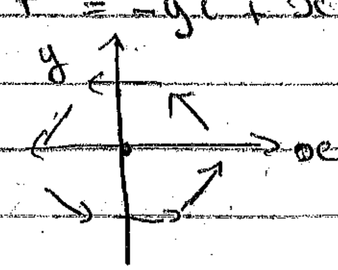
$\vec{F} = \langle x, y \rangle$ curl $\vec{F} = 0$ no twist at a given point

Example:



$\vec{F} = \langle a, b \rangle$ constant field then curl $\vec{F} = 0$

If $\vec{F} = -y\vec{i} + x\vec{j} = \vec{V}$ vector field.



curl $\vec{F} = N_x - M_y = 1 - (-1) = 2$

curl $\vec{F} = 2\omega$

with $\omega = 1$ ($|\vec{V}| = r\omega$)

In Physics.

Force → $\dfrac{\vec{F}}{m} = \dfrac{d\vec{V}}{dt}$ ← vector velocity | translation

inertia — curl — curl

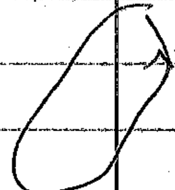
$\dfrac{\text{Torque}}{\text{moment of inertia}} = \dfrac{d\vec{\omega}}{dt}$ | rotation

curl$(\vec{F})$ = torque per unit of moment of inertia.
tells you how quickly the angular velocity is increasing.

Green theorem:

$$\text{curl}(\vec{F}) = N_x - M_y$$

If the field is not conservative $N_x \neq M_y$ we can still find a way to compute line integral in 2D



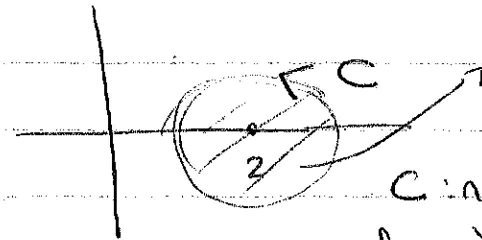
C

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y)\, dA$$

or $$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA$$

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}}$$

Example.



Area A

C: radius of 1, density of mass = 1

Find $\oint_C \underbrace{y e^{-x}}_{M} dx + \underbrace{\tfrac{1}{2} x^2 + e^{-x}}_{N} dy$

Green th. $\oint_C = \iint_R (N_x - M_y)\, dA = \iint_R (x \cancel{e^{-x}} - \cancel{x} e^{-x} \cancel{+} e^{-x}$

$\oint_C \vec{F} \cdot d\vec{r} = \iint_R x \, dA = \bar{x} \times \text{Area}(A) = \pi \times 2$

$\bar{x}$ ↳ center of Mass