

## Dot Product and Cross Product

The reader should recall the availability of vector and matrix arithmetic including addition of vectors or matrices, multiplication of vectors or matrices by scalars, multiplication of vectors by matrices, multiplying matrices by other matrices, and taking the determinant of a square matrix. The definition and basic properties of these operations will not be covered here and will be employed with little comment. We will use the standard definitions of *dot product* and *cross product* as follows. If  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ , the scalar-valued dot product of  $\mathbf{u}$  and  $\mathbf{v}$  is given by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}_x \mathbf{v}_x + \mathbf{u}_y \mathbf{v}_y + \mathbf{u}_z \mathbf{v}_z.$$

The vector-valued cross product of  $\mathbf{u}$  and  $\mathbf{v}$  is given by

$$\mathbf{u} \times \mathbf{v} = \left[ \begin{array}{ccc} (\mathbf{u}_y \mathbf{v}_z - \mathbf{u}_z \mathbf{v}_y) & (\mathbf{u}_z \mathbf{v}_x - \mathbf{u}_x \mathbf{v}_z) & (\mathbf{u}_x \mathbf{v}_y - \mathbf{u}_y \mathbf{v}_x) \end{array} \right].$$

Any of the following properties of the dot product and cross product that are not immediately obvious can be proven directly from the definitions.

- (a)  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$ .
- (b)  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ .
- (c)  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ .
- (d)  $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{w})$ .
- (e)  $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$  and  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ .
- (f)  $(\mathbf{u} \times \mathbf{u}) = \mathbf{0}$ .
- (g)  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w}$ .
- (h)  $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})) \mathbf{u}$ .
- (i) For any vector  $\mathbf{w}$  there exist vectors  $\mathbf{u}$  and  $\mathbf{v}$ , perpendicular to each other and both perpendicular to  $\mathbf{w}$ , such that  $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ ,  $\mathbf{v} \times \mathbf{w} = \mathbf{u}$ , and  $\mathbf{w} \times \mathbf{u} = \mathbf{v}$ .
- (j) For all vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ ,  $\mathbf{w} \neq \mathbf{0}$ , if  $\mathbf{u} \times \mathbf{w} = \mathbf{v} \times \mathbf{w}$  and  $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ , then  $\mathbf{u} = \mathbf{v}$ .